BLOOM

$\boxed{20^{54}}$

| 1 | 2 | 3 | 4 | 5 | Σ |
|---|---|---|---|---|---|
| 1 | 1 | 0.8 | 1,15 | 0,95 | 4.9 |

1. Not necessarily. As a counterexample take $\mathbb{R}^2$ with the discrete metric, $d(x,y) = 0$ if $x = y$
$d(x,y) = 1$ if $x \neq y$

Let $x$ be $(-2,0)$ and $r = 2$
Then $B^d_r(x) = \{a \in \mathbb{R}^2 \mid d(x,a) < 2\} = \mathbb{R}^2$
Let $y$ be $(3,0)$ and $s = 3$
Then $B^d_s(y) = \{a \in \mathbb{R}^2 \mid d(y,a) < 3\} = \mathbb{R}^2$

It is clear that in this case $B^d_r(x) = B^d_s(y)$
but $x \neq y$ and $r \neq s$.

2. A1. $d^{(1)}(x,y) \geq 0$ since $d^{(1)}$ is a metric
$d^{(2)}(x,y) \geq 0$ since $d^{(2)}$ is a metric
Therefore, $\max(d^{(1)}(x,y), d^{(2)}(x,y)) \geq 0$

Furthermore, if $d(x,y) = 0$, $\max(d^{(1)}(x,y), d^{(2)}(x,y)) = 0$
so $d^{(1)}(x,y) = 0$ and $d^{(2)}(x,y) = 0$ and since $d^{(1)}$ and $d^{(2)}$ are both metrics this implies $x = y$.
Also, clearly, when $x = y$, $d(x,y) = 0$.

A2. Since $d^{(1)}$ and $d^{(2)}$ are metrics we have
$d^{(1)}(x,y) = d^{(1)}(y,x)$ and $d^{(2)}(x,y) = d^{(2)}(y,x)$

$d(x,y) = \max(d^{(1)}(x,y), d^{(2)}(x,y)) = \max(d^{(1)}(y,x), d^{(2)}(y,x))$
$= d(y,x)$

so $d(x,y) = d(y,x)$.

A3. $\max(d^{(1)}(x,z), d^{(2)}(x,z)) \overset{?}{\leq} \max(d^{(1)}(x,y), d^{(2)}(x,y)) + \max(d^{(1)}(y,z), d^{(2)}(y,z))$

Suppose $d^{(1)}(x,z) = \max(d^{(1)}(x,z), d^{(2)}(x,z))$
Since $d^{(1)}$ is a metric we have
$d^{(1)}(x,z) \leq d^{(1)}(x,y) + d^{(1)}(y,z)$
$\leq \max(d^{(1)}(x,y), d^{(2)}(x,y)) + \max(d^{(1)}(y,z), d^{(2)}(y,z))$

If $d^{(2)}(x,z) = \max(d^{(1)}(x,z), d^{(2)}(x,z))$
a similar argument holds.

This shows that $d(x,z) \leq d(x,y) + d(y,z)$.

$\checkmark$

This means all three axioms are satisfied so $d$ is always a metric on $X$.

3. $A$ is connected so any continuous function $f: A \to \{0, 1\}$ is constant

$\pm$

Assume for contradiction $B$ is disconnected. This means there exists a continuous function $f: B \to \{0,1\}$ and $b_1, b_2 \in B$ such that $f(b_1) = 0$ and $f(b_2) = 1$.

$b_1 \in B \Rightarrow b_1 \in \bar{A} \Rightarrow B_\varepsilon(b_1) \cap A \neq \phi \quad \forall \varepsilon > 0$
since $f$ is continuous $f(c) = 0 \quad \forall c \in B_\varepsilon(b_1)$ for $\varepsilon < 1$ ? Why?
Therefore there is an $a_1 \in B_\varepsilon(b_1) \cap A$ such that $f(a_1) = 0$

$b_2 \in B \Rightarrow b_2 \in \bar{A} \Rightarrow B_\varepsilon(b_2) \cap A \neq \phi \quad \forall \varepsilon > 0$
since $f$ is continuous $f(d) = 1 \quad \forall d \in B_\varepsilon(b_2)$ for $\varepsilon < 1$
Therefore there is an $a_2 \in B_\varepsilon(b_2) \cap A$ such that $f(a_2) = 1$

However, this contradicts that any continuous function $A \to \{0,1\}$ is constant, since $f_{1A}$ is continuous and $f_{1A}(a_1) = 0 \neq 1 = f_{1A}(a_2)$.
Therefore, $B$ is connected.

4. 4 Definition: A subset $A$ of a Hausdorff space $X$ is compact if every open cover for $A$ has a finite subcover.

Let $C$ be an open cover for $A \cap B$.
$A$ is compact so $A$ is closed so $X \setminus A$ is open.
Therefore $C \cup X \setminus A$ is an open cover for $B$.

If $C$ does not have a finite open subcover for $A \cap B$, this means $C \cup X \setminus A$ does not have a finite open subcover for $B$, but this contradicts that $B$ is compact.

Therefore, $C$ has a finite open subcover, so $A \cap B$ is compact.
The empty set is compact, so if $A = \phi$ or $B = \phi$, $A \cap B$ is compact.

5. The equation $9x = 1 - x^5$ has a solution when $f(x) = 0$ for $f(x) = 1 - x^5 - 9x$

$f$ is continuous. $f(0) = 1$ and $f(1) = -9$
$f(1) < 0 < f(0)$ so the intermediate value theorem guarantees there exists a $c \in [0,1]$ such that $f(c) = 0$

Furthermore, $f'(x) = -5x^4 - 9 < 0 \quad \forall x \in \mathbb{R}$
This means $f$ is strictly decreasing and therefore our solution $c$ is unique.
explain.

Use material from this course for an exhaustive proof.

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